Open Problems

In

Number Theory

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A REMARK

These are 233 open problems and conjectures in number theory. The list contains problems from the three known parts of the discipline : Elementary, Algebraic and Analytic. As usual, open questions in algebraic number theory may have links with other *algebraic* subjects, like algebraic topology and algebraic geometry.

I am not pretending that the list is exhaustive. Some problems are as old as the Greek mathematics and others have been posed quite recently. Two of the problems are among the *Clay Institute Millennium Problems*.

One has to note, surprisingly, that a lot of not yet proved results in number theory depend on two big problems in Mathematical Analysis: the *Riemann hypothesis* or the *generalized Riemann hypothesis*. Also, there are many consequences of the ABC conjecture in elementary number theory.

The problems * with the sign $(*)$ are mine. I think that they are quite interesting and, as far as I know, I have not found them in the literature. The origin of this compilation is in my draft *Number theory: ideas and concepts* (2018). Any mistakes that have remained are mine.

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...A mathematician, in what follows, will be then defined as somebody who has published at least the demonstration of a non-trivial theorem.

Jean Dieudonné

THE PROBLEMS

1 The problem of finding Egyptian fractions[1](from Antiquity) : Does a greedy algorithm always succeed to find an Egyptian fraction with odd denominators ?

2 The nature of the integers $x1$ $, x_2, \dots, x_k$ (max of x_k, $max(x_i - x_{i-1})$,...) in an Egyptian representation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k} = 1$$

3 Is there an efficient algorithm to find an Egyptian representation for a rational number $\frac{a}{b}$?

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[1] A *unit* fraction is of the form $\frac{1}{q}$, where $q \in \mathbb{N}^*$. An Egyptian fraction $\frac{a}{b}$ is the finite sum of unit fractions : $\frac{a}{b} = \sum_1^n \frac{1}{q_i}$, where q_i are distinct. So $\frac{1}{2} + \frac{1}{2}$ is not an Egyptian fraction, but $\frac{1}{2} + \frac{1}{3}$ is. It was in this way that Egyptians wrote factions (As for example in the **Rhind** papyrus) except for $\frac{2}{3}$. In 2015 Chinese Sun **Zhiwei**$(1965 -)$ conjectured that every $r = \frac{a}{b} \in \mathbb{Q}^+$ has an Egyptian fraction representation such that b is a *practical* number, a conjecture proved by American David **Eppstein**$(1963-)$ in 2021. See also the **Erdös–Straus** conjecture.

4 Are there infinitely many amicable[2] numbers ? (from Antiquity).

5 Are there infinitely many even[3] perfect numbers ? (from Antiquity).

6 Is there any odd[4] perfect number ? (from Antiquity).

7 Infinitude of the primes of the form[5] : $p\,\#= 2\times 3\times 5\times 7\ldots\times p+1$ (**Euclid**).

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[2] The Greeks were haunted by the *positive* integers: $1,2,3,\ldots$ and their properties. They have transmitted us the study of : prime, perfect, and amicable numbers. Other numbers they studied have geometrical flavor, they are called *figurative* numbers like: triangular, pyramidal, and general polygonal numbers. Two positive integers are *amicable* if each one is the sum of the divisors of the other, $(220,284)$ is a pair of amicable numbers. **Thabit ibn Qurrah**$(826-901)$ was the first to give a rule generating some of them.

[3] A perfect number equals to the sum of its proper divisors like $6=1+2+3$. The conjecture is equivalent to show that there are infinitely many Mersenne numbers M_p. **Euler** completed the proof of the fact that N is an even perfect integer iff $N=2^{n-1}(2^n-1)$.

[4] Unlike even perfect numbers, number theorists think that there are no odd perfect numbers at all, let alone an infinity of them.

[5] The product of the n first primes is called primordial of p_n and noted $p_n\,\#$ or simply $p\,\#$. Any number of the form $p\,\#+1$ is called a *Euclid number*. See also the Euclid –Mullin problem(1963).

8 Are there squares in the sequence : $p\,\# = 2 \times 3 \times 5 \times 7 \ldots \times p + 1$?

9 The Congruent number problem : How can we determine that a given positive integer is a congruent number[6]? (Probably from Antiquity).

10 The Fibonacci primes $(120?)$: Are there infinitely many primes in the sequence of Fibonacci numbers[7] $\mathcal{F}_n$?

11 The Fibonacci problem $(120?)$: Let $\mathcal{F}_n$ be the sequence of Fibonacci numbers, find the value[8] of :

$$\lim_{n\to+\infty} \frac{\mathcal{F}_{n+1}}{\mathcal{F}_n}$$

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[6] A positive integer is *congruent* if it equals to the area of a right triangle. It is known that the solution of the Birch–Swinnerton–Dyer conjecture gives a solution for the congruent problem via Tunnell's theorem.

[7] French Édouard **Lucas**$(1842 - 1891)$showed in 1876 that $\sum_1^n \mathcal{F}_i = \mathcal{F}_{n+2} - 1$.

[8] $\frac{\mathcal{F}_{n+1}}{\mathcal{F}_n} = \frac{1}{1+\frac{1}{1+\frac{1}{1+\cdots}}}$. It is evident that the limit is bounded by 2. About Fibonacci numbers, see also the Wall–Sun–Sun problem.

12 The Mersenne primes[9] $(164?)$: Are there infinitely many Mersenne primes $M_p = 2^p - 1$?

13 Are there infinitely many composite[10] Mersenne numbers $M_p = 2^p - 1$?

14 Is every Mersenne number $M_p = 2^p - 1$ square−free ?

15 Which double[11] Mersenne numbers $2^{2^p-1} - 1$ are prime ?

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[9] It is easy to show that if $2^n - 1$ is prime then n itself is prime. It is for this reason that Mersenne primes are usually written : $M_p = 2^p - 1$.

[10] All numbers $2^n - 1$ when n is composite are composite, but the numbers $2^p - 1$ can be either composite or prime.

[11] A double Mersenne prime is of the form : $2^{2^p-1} - 1$. It is prime only if $2^p - 1$ is prime.

16 There are finitely[12] many Fermat primes : $F_n = 2^{2^n} + 1$.

17 Are there infinitely many composite Fermat numbers : $F_n = 2^{2^n} + 1$?

18 Are there infinitely many square−free Fermat numbers : $F_n = 2^{2^n} + 1$?

19 Are there infinitely many primes[13] verifying : $2^{p-1} \equiv 1\ [p^2]$?

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[12] In 1640, French Pierre de **Fermat**$(1607 - 1665)$ stated wrongly that all F_n are primes. In 1844 , German mathematician Ferdinand **Eisenstein**$(1823 - 1852)$ conjectured that there are infinitely many of them. Today, number theorists think the opposite.

[13] In 1640, French Pierre de **Fermat**$(1607 - 1665)$ stated and Leonhard **Euler**$(1707 - 1783)$ proved in 1736, that if $(a, b) = 1$ then $a^{p-1} \equiv 1[p]$, a result known as *Fermat little theorem*. So in this case, there are infinitely many primes that verify the equation $2^{p-1} \equiv 1\ [p]$.

20 Are there infinitely many square pseudoprimes[14] ?

21 The numbers $\zeta(2k+1)(1737)$: Let $k \geq 2$. The nature[15] (at least irrationality) of the numbers[16] :

$$\zeta(2k+1) = \sum_{n=1}^{+\infty} \frac{1}{n^{2k+1}}$$

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[14] Fermat little theorem says that if $(a, b) = 1$ and p is a prime then $a^{p-1} \equiv 1[p]$. A *pseudoprime* to base a is an odd composite integer n such that $a^{n-1} \equiv 1[n]$. If the previous relation is true for every prime a, then n is a *Carmichael number*.

[15] A real number is *rational* $r \in \mathbb{Q}$ if $r = \frac{a}{b}$ with $a, b \in \mathbb{Z}$, otherwise it is *irrational*. A complex number α is *algebraic* if is the solution of a polynomial equation of degree : $\sum_{k=0}^{k=n} a_k x^k = 0$, a number which is not algebraic is *transcendental*. Every transcendental number is necessary irrational but the opposite is not true; there are irrationals that are algebraic.

[16] Swiss Leonhard **Euler**$(1707-1783)$ proved around 1737 that $\zeta(2k) = \frac{(2\pi)^{2k}(-1)^{k+1}B_{2k}}{2(2k)\,!}$ and we know that they are all irrational. The sums $\zeta(2k+1) = \sum \frac{1}{n^{2k+1}}$ are more difficult to handle. In 1978! French mathematician Roger **Apéry** $(1916-1994)$ astonished number theorists by proving that $\zeta(3)$ is irrational. However, in 2001 , another French, Tanguy **Rivoal** proved that there is an infinity of irrationals in the sequence $\zeta(2k+1)$ without determining which ones. $\zeta(0)$ and all $\zeta(1-2k)$ are rational. It is easy to show that the sums $\sum_{n\geq 0} \frac{1}{(2n+1)^{2k}}$ are irrational for all $k \geq 1$. The sums $\sum_{n\geq 0} \frac{1}{(2n+1)^{2k+1}}$ are irrational for infinitely many k.

22 The nature(at least irrationality) of the Euler−Mascheroni constant[17]:

$$\gamma = \lim_{n\to+\infty}\left(\sum_{1}^{n}\frac{1}{k} - log(n)\right)$$

23 The Euler numbers[18] E_n problem : If $p = 8k + 1$ does :

$$p \nmid E_{\frac{p-1}{2}} \ ?$$

24 The strong Goldbach conjecture[19] (1742).

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[17] Attributed to Swiss Leonhard **Euler**(1707 − 1783) and Italian Lorenzo **Mascheroni**(1750 − 1800). The constant is also defined as :

$$\gamma = lim_{z\to1}(\zeta(z) - \frac{1}{z-1})$$

or by the integral

$$\gamma = -\int_{0}^{+\infty}\frac{log(x)}{e^{x}}dx$$

[18] These numbers occur as coefficients in the infinite series $sec(x) = \frac{1}{\cos(x)} = \sum_{n\geq0}\frac{E_{2n}(-1)^{n}}{2n!}x^{2n}$.

[19] This conjecture was the 8th Hilbert's problem. It saw light in 1742, in a letter of Christian **Goldbach**(1690 − 1764) to Leonhard **Euler**(1707 − 1783). There is also the weak Goldbach conjecture : every odd integer $n \geq 7$ is the sum of three primes. In 1937, Soviet mathematician Ivan **Vinogradov**(1991 − 1983) proved that the weak Goldbach conjecture is true for every sufficiently large odd integers. In 1997 French Jean-Marc **Deshouillers**(1946−) *et al*. showed that, assuming the generalized Riemann hypothesis, the weak Goldbach conjecture is true. The conjecture has been proved by Harald **Helfgott**(1977 −) in 2013. Every even integer ≥ 3 is the sum of a prime and an odd integer. It is easy to show that the strong Goldbach conjecture implies the weak one. The weak Goldbach conjecture implies that every even integer $n \geq 4$ is the sum of four primes. The strong

25 Are there infinitely many primes of the form $n^2 - 2$?

26 The Euler sums conjecture[20] (1752) : Are there positive integers (a, b, c, d, e, f, g) such that

$$a^6 + b^6 + c^6 + d^6 + e^6 + f^6 = g^6 \quad ?$$

Are there positive integers $(a, b, c, d, e, f, g, h, i, j)$ such that

$$a^9 + b^9 + c^9 + d^9 + e^9 + f^9 + g^9 + h^9 + i^9 = j^9 \quad ?$$

27 The Waring[21] problem(1770) : The value of the function $g(k)$.

Goldbach conjecture with the *twin primes conjecture* are among the most known and difficult problems in prime number theory.

[20] After proving *Fermat Last Theorem* for $n = 3$ (the proof was after all incorrect, see Harold **Edwards** FLT) Leonhard **Euler**$(1707 - 1783)$ conjectured that if $\sum_{i=1}^{n} a_i^k = b^k$ then $n > k$. The conjecture turned to be false, many example were found like $k = 4, 5, 7, 8$. See also the Lander–Parkin–Selfridge conjecture.

[21] By English Edward **Waring**$(1736 - 1798)$: Every integer n is the sum of a $g(n)$ of m^{th} -powers , this means equivalently that $g(m) = k$. For example in the same year 1770, **Lagrange**$(1736 - 1813)$ proved that every integer is the sum of four squares so : $g(2) = 4$. Many results since then are known: $g(3) = 9$, $g(4) = 19$, $g(5) = 37$. No general formula is proved for all integers.

28 The Euler conjecture on the Waring problem :

$$g(k) = 2^k + \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor - 2.$$

29 The Wilson[22] primes (1770) : Are there infinitely many Wilson primes[23]?

30 The Lagrange conjecture (1775) : every odd natural integer $n > 5$ is of the form $p + 2q$, where p and q are primes.

31 The Euler idoneal conjecture (1778) : There are only 65 idoneal[24] numbers.

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[22] By English judge and mathematician John **Wilson**$(1741-1793)$, he was a student of Edward **Waring**$(1736-1798)$.

[23] They are primes p such that : $p^2|(p-1)!+1$. Every prime p divides $(p-1)!+1$, a result known as Wilson's theorem, it was known to Muslim scientist **Hasan Ibn al-Haytham** (ابن الهيثم)$(966-1039)$.

[24] An integer D is idoneal if : $D \neq ab+bc+ac$, where a, b and c are distinct positive integers.

32 Does a perfect Euler cuboid[25] exist?

33 Is there a triangle[26] with integer sides, medians and area?

34 The **Legendre**[27] conjecture(17??): There are always primes p such that:

$$n^2 < p < (n+1)^2$$

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[25] An Euler cuboid is a special case of *Euler bricks*. The solution of this problem amounts to a solution of as system of Diophantine equations:

$$\begin{cases} x^2 + y^2 = w^2 \\ x^2 + z^2 = t^2 \\ y^2 + z^2 = s^2 \\ x^2 + y^2 + z^2 = r^2 \end{cases}$$

[26] A triangle with integer sides and area is called a *Heron triangle*, after Greek mathematician **Heron**. There are infinitely many Heron triangles, with rational sides and areas.

[27] Attributed to French mathematician Andrien–Marie **Legendre**(1752 – 1833) , he probably stated in his famous book "*Essai in the Theory of Numbers*". It was proposed by German mathematician Edmund **Landau**(1877 – 1938) at the International Congress of Mathematicians(1912).

35 Let n be a positive integer. Is there a formula[28] that gives primitive roots modulo a positive integer n ?

36 Every sufficiently large integer is the sum of 4 nonnegative cubes.

37 Every sufficiently large integer is the sum of 3 cubes.

38 The Gauss conjecture[29] on primitive roots(1801).

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[28] We say that a is a *primitive root modulo n* , if for every $(n,b)=1$, there exist k such that $a^k \equiv b[n]$. If n is prime then there are always primitive roots modulo n, if n is composite it is not necessarily true. The number of primitive roots modulo n is $\varphi(\varphi(n))$, where φ is the Euler totient function.

[29] Artin conjecture is more general.

39 What are the factorial[30] domains $\mathbb{Q}[\sqrt{d}]$ when d >0?

40 The Gauss conjecture(1801) : There are infinitely many quadratic number fields $K = \mathbb{Q}(\sqrt{d})$ with class number $h_K = 1$.

41 The class number problem(1801) : For every class number h, the number of quadratic number fields with h is finite.

42 Are there infinitely many factorial domains ?

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[30] The genius German Carl **Gauss** $(1777-1855)$ has made an astonishing conjecture in 1801 , namely that the only imaginary quadratic factorial domains (i.e. with $h = 1$) correspond to the cases : $D = -1, -2, -3, -7, -11, -19, -43, -67, -167$. German Kurt **Heegner** $(1893-1965)$ proposed a proof which contained a flaw(1952). In 1967, Harold **Stark** $(1939-)$, corrected **Heegner**'s proof. In 1966, English Alan **Baker**$(1893-1965)$ proved it by another method. Another conjecture by **Gauss** was that $lim_{D\to-\infty} h(D) = -\infty$, proved by German Hans **Heilbronn** $(1908-1975)$ in 1934.

43 The problem of finding the ring of integers for non−quadratic fields.

44 The Gauss circle problem : How many lattice points[31] are there inside a plane circle with origin$(0,0)$ and radius r?

45 The Germain primes[32] (1825) : Let p be a prime, are there infinitely many primes of the form $2p+1$?

46 Let p be a prime, are there infinitely many primes of the form $2p-1$?

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[31] Lattice points are points ω in $\mathbb{R}^2$ with integer coordinates $\omega(a,b)$.

[32] Attributed to French mathematician Sophie **Germain**$(1776-1831)$. Before **Germain**, Leonhard **Euler**$(1707-1783)$ has proved that if $p=4n+3$, then $2p+1$ is prime iff $2^p\equiv 1[2p+1]$. A related concept is the *Cunningham chain* , which is a sequence of primes of the form $a_i=2p+1$ such that $a_{i+1}=2a_i+1$. This was introduced by Alan **Cunningham**$(1842-1928)$.

47 The Fermat−Catalan conjecture[33](1844) : Let (a, b, c) integers and (x, y, z) positive coprime integers. The equation

$$x^a + y^b = z^c$$

has a finite number of integer solutions such that : $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$.

48 The Hermite problem[34](1848) : The possibility of representing a real number by a sequence of natural integers.

49 Does any general polynomial[35] $ax^2 + bx + c\ (a \neq 0)$ generate infinitely many primes ?

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[33] It is known that the ABC conjecture $\Longrightarrow$ the Fermat−Catalan conjecture. Éugène **Catalan**$(1814 - 1894)$ was a French mathematician.

[34] Due to French Charles **Hermite**$(1822 - 1901)$.

[35] See for example the book of Godfrey **Hardy** $(1877 - 1947$*: An Introduction to the Theory of Numbers* $(p. 23)$. In a pioneering work, German Johann **Dirichlet**$(1805 - 1859)$ proved that, under certain conditions, the polynomial $ax^2 + bxy + cy^2$ contains infinitely many primes.

50 The Polignac conjecture[36](1848) : $\forall n \in \mathbb{N},\ \exists\, p_1, p_2$ such that :

$$n = p_2 - p_1$$

51 The problem of gaps between primes :

$$\Delta p = p_n - p_{n-1}$$

52 Dirichlet divisor problem(1849) : Let $d(n)$ be the number of divisors of an integer n. The problem is to estimate δ in:

$$D(n) = \sum_{k=1}^{n} d(k) = n log(n) + n(2\gamma - 1) + \delta.$$

γ is the Euler–Mascheroni constant.

53 The Kummer–Vandiver[37] conjecture(1849).

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[36] n is evidently even. It was proposed by French Alphonse de **Polignac**$(1826 - 1863)$, in fact the conjecture states that there is an infinity of such (p_1, p_2). The case $n = 2$ corresponds to the twin primes conjecture.

[37] By German Ernest **Kummer**$(1810 - 1893)$, and rediscovered later by American Harry **Vandiver**$(1882 - 1973)$.

54 Are there infinitely many regular[38] prime numbers?

55 The Pollock conjecture[39] I (1850) : Every positive integer is the sum of at most 5 tetrahedral numbers.

56 The Pollock conjecture II (1850) : Every positive integer is the sum of at most 13 icosahedral numbers.

57 The Pollock conjecture III (1850) : Every positive integer is the sum of at most 21 dodecahedral numbers.

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[38] We know since 1915, that there are infinitely many *irregular* primes. A prime p is regular if it does not divide the numerators of the Bernoulli numbers B_0 , $B_2, \dots, B_{p-3}$, or it does not divide the class number of $\mathbb{Q}(\zeta_p)$. See also Siegel conjecture. Regular primes were introduced by Ernest **Kummer**$(1810-1893)$ and he proved *Fermat Last Theorem* for all regular primes.

[39] The three conjecture have been posed by English lawyer Jonathan **Pollock**$(1783-1870)$.

58 The Liouville numbers(1851) : Are the numbers[40] $\sum_{n\geq 1} k^{-n^2}$, transcendental for $k \geq 2$?

59 Is e^{π} a Liouville number[41] ?

60 The Prouhet problem[42] (1851).

61 The Bunyakovsky conjecture[43](1857) : A criterion of three conditions for a polynomial to give an infinity of primes.

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[40] Considered by French mathematician Joseph **Liouville**$(1809-1882)$ in his famous paper of $1844-1851$.

[41] A Liouville number is a number that can be approximated in a certain way, following an idea introduced by French mathematician Joseph **Liouville**$(1809-1882)$. The number $\sum_{n\geq 1} 10^{-n!}$ is a Liouville number. It is known that the numbers π , e and $log(2)$ are not Liouville numbers.

[42] Attributed to French mathematician Eugène **Prouhet**$(1817-1867)$. It was subsequently studied by French mathematician Gaston **Tarry**$(1843-1913)$ and American mathematician Edward **Escott**$(1868-1946)$.

[43] By Russian mathematician Victor **Bunyakovsky**$(1804-1889)$. He is also known for the inequality we call today Cauchy—Schwarz *inequality*.

62 The Riemann hypothesis[44] (1859).

63 The zeros of the zeta function (1859) : The amount of the roots of the classical zeta function on the critical line $[0,T]$ is asymptotic to

$$\frac{T}{2\pi} log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi}$$

64 The zeros of the zeta function (1859) : Are all the non$-$trivial roots of the zeta function ζ simple ?

65 The zeros of the zeta function (1859) : The nature(at least irrationality) of the numbers γ_n, where

$$\zeta\left(\frac{1}{2} + i\gamma_n\right) = 0.$$

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[44] The *Riemann hypothesis* stated by German mathematician Bernhard **Riemann**$(1826 - 1866)$ in his seminal paper of 1859 is considered by many as the greatest unsolved problem in the whole of mathematics.

66 The Wolstenholme problem[45](1862) : If

$$\binom{2n-1}{n} \equiv 1 \ [n^3]$$

then n is a prime number.

67 The Brocard problem[46](1876) : Let the integer > 7 . Are there any squares of the form $k^2 = n\,! + 1$?

68 The Brocard conjecture : There are always 4 primes[47] between two squares of consecutive primes.

69 The Catalan–Mersenne number[48](1876) : Is C_5 prime ?

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[45] English Joseph **Wolstenholme**$(1829-1891)$ proved the converse of the conjecture in 1862. See McIntosh conjecture.

[46] By French Henri **Brocard**$(1845-1922)$. The number $n!+1$ is a square for $n = 4, 5$ and 7. It is easy to show that $n!-1$ cannot be a square.

[47] Which means : $\pi({p_{n+1}}^2) - \pi({p_n}^2) \geq 4$. In its stronger form : between every two consecutive odd squares there are always 5 primes.

[48] A Mersenne prime is of the form $M_p = 2^p - 1$. Consider now the sequence : $2^{p_1} - 1 = p_2$, $2^{p_2} - 1 = p_3$ .... A Catalan–Mersenne prime is of the form $C_n = 2^{C_{n-1}} - 1$.

70 The Catalan−Mersenne conjecture[49] (1876) : Are there infinitely many primes of the form C_n?

71 The Oppermann conjecture[50] (1877) : There are at least two primes between $x(x-1)$ and $x(x+1)$.

72 The nature (at least irrationality) of Catalan constant[51] (1865):

$$C = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)}$$

73 The Lucas primes (1878) : Are there infinitely many primes in the Lucas sequence[52] $\mathcal{L}_n$?

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[49] By French Eugène **Catalan** $(1814-1894)$. The Catalan−Mersenne numbers are special case of double Mersenne numbers $M_{M_p} = 2^{2^p-1}-1$.

[50] Which means : $\pi(x^2+x) > \pi(x^2) > \pi(x^2-x)$. By Danish Ludvig **Oppermann** $(1817-1883)$.

[51] The constant is due to French Éugene **Catalan** $(1814-1894)$.

[52] Defined by French Édouard **Lucas** $(1842-1891)$ as $\mathcal{L}_0 = 1, \mathcal{L}_1 = 3$ and $\mathcal{L}_{n+1} = \mathcal{L}_n + \mathcal{L}_{n-1}$: $1, 3, 4, 7, 11, 18, 29, \ldots$

74 The Lucas problem (1878) : Let $\mathcal{L}_n$ be the sequence of Lucas numbers, find the value[53] of :

$$\lim_{n\to+\infty} \frac{\mathcal{L}_{n+1}}{\mathcal{L}_n}$$

75 The Dedekind conjecture on the Dedekind L –functions[54] : $\zeta_K(z)/\zeta(z)$ is entire.

76 The Lemoine conjecture (1895) : $\forall n$ an odd integer > 5 :

$$n = prime\ + semiprime$$[55].

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[53] It is evident that the limit is bounded by 2 and :

$$\frac{\mathcal{L}_{n+1}}{\mathcal{L}_n} = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$$

[54] By German Richard **Dedekind**$(1831-1916)$. It is a special case of Artin's conjecture.

[55] A semiprime equals to the product of two primes like $21 = 3 \times 7$. The conjecture was made by French Emile **Lemoire**$(1840-1912)$.

77 The Pierpont's primes[56] conjecture(1895) : There are infinitely many Pierpont primes.

78 The 9^{th} Hilbert problem (1900) : The reciprocity law in number fields (the non-**Abelian** case).

79 The twin[57] primes conjecture : Are there infinitely many twin primes (p_n, p_{n+1}).

80 The number[58] $T(n)$ of twins prime $\leq n$ is:

$$T(n) \sim \alpha \int_2^n \frac{dx}{log^2(x)}$$

$\alpha = 1.320323 \ldots$

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[56] A Pierpont prime is a prime of the form $2^a \times 3^b + 1$. It was posed by American James **Pierpont**$(1866 - 1938)$.

[57] This was 8^{th} Hilbert's problem. In 1900 , known German mathematician David **Hilbert**$(1862 - 1943)$ posed his famous list of 23 problems, the list has shaped mathematical research in the 20^{th} century. The sum of twins' reciprocals $\sum_p (\frac{1}{p_n} + \frac{1}{p_{n+1}})$ converges to ***Brun*** *constant* C_B, named after Norwegian mathematician Viggo **Brun**$(1885 - 1978)$ who proved the convergence of the series in 1915 . We have an interesting propriety of these twin primes due to American mathematician Paul **Clement** in 1947, which is similar to **Wilson** theorem: the integers n and $n + 2$ are twin primes iff : $n(n + 2) \mid 4(n - 1)! + 4 + n$.

[58] Known as the Hardy$-$Littlewood conjecture, made in 1923. It has a more general form.

81 Are there always twin primes in the interval $[2, 2n]$?

82 Infinitude of primes of the form[59] : $n\,! - 1$.

83 Infinitude of primes of the form : $n\,! + 1$.

84 Infinitude of twin primes of the form[60] : $(\ n! - 1\ ,\ n! + 1\ )$.

85 Infinitude of the Germain twin primes : $(\ 2p - 1\ ,\ 2p + 1\ )$.

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[59] Called factorial primes.

[60] Since the numbers $n! \pm k$ are composite for $2 < k \leq n$, it is highly probable that the numbers $n! + 1$ and $n! - 1$ could be prime, and perhaps for infinitely many times.

86 Are there infinitely many primes of the form[61] : $p\,\# - 1$?

87 Are there infinitely many primes of the form : $p\,\# + 1$?

88 Are there infinitely many twin primes of the form : $p\,\# \pm 1$?

89 The Schinzel conjecture[62] : There is always a prime between x and $log^2(x)$.

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[61] The primes of the form $p\,\# - 1$ and $p\,\# + 1$ are called *primordial* primes. We know that primordial of p is as mentioned before : $p\,\# = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times \ldots \times p$.

[62] By Polish mathematician Andrzej **Schinzel**$(1937 - 2021)$. The Schinzel conjecture implies the Fortune conjecture.

90 The infinitude of twin quadruplets[63].

91 The Hilbert–Polya conjecture[64] $(1900 - 1912)$: The nontrivial roots of the function ζ are the eigenvalues of a linear operator A of the form :

$$A = A^{*}.$$

92 The Lindelöf hypothesis[65](1901) .

93 The Grand Lindelöf Hypothesis (GLH).

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[63] Like : $11 - 13 - 17 - 19$.

[64] A^{*} stands for the self-adjoint operator of A. In the case of square matrices, it is similar to the Hermitian matrix : a matrix A verifying : $\overline{A^{T}} = A$. George **Pólya**$(1887 - 1985)$ was a Hungarian mathematician.

[65] By Finnish Ernest **Lindelöf**$(1870 - 1946)$. We know that : the Riemann hypothesis $\Longrightarrow$ the Lindelöf hypothesis.

94 The generalized Riemann hypothesis[66] (GRH).

95 The Dickson[67] conjecture (1904).

96 Are there infinitely many prime pairs $(p, 2p-1)$?

97 Are there infinitely many prime pairs $(p, 2p+1)$?

98 Cullen[68] primes (1905) : Are there infinitely many Cullen primes ?

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[66] Also known as the Grand Riemann Hypothesis (GRH).

[67] By American Eugene **Dickson** $(1874-1954)$. The famous *Dirichlet theorem* on primes in arithmetic progressions is a special case.

[68] By Irish priest James **Cullen** $(1867-1933)$. A Cullen prime is of the form $n \times 2^n + 1$. A Woodall number is of the form $n \times 2^n - 1$.

99 The Carmichael[69] conjecture (1907) : For every positive integer n_1, there is at least a positive integer n_2 such that $\varphi(n_1) = \varphi(n_2)$.

100 The Wieferich[70] primes(1909) : Are there infinitely many primes p such that :

$$p^2 \mid 2^p - 2 \qquad ?$$

101 The Landau conjecture (1912) : Are there infinitely[71] many primes of the form : $n^2 + 1$.

102 The number of primes the form $n^2 + 1$ is

$$L(n) \;\sim\; \alpha \int_2^n \frac{dx}{log(x)}$$

$\alpha = 0.686406 \ldots$

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[69] The function φ is the Euler totient function, it give the number of integers $< n$ and comprime to n. The conjecture is by American mathematician Robert **Carmichael**$(1879 - 1967)$. This is a bit different from : are there infinitely many couples (n_1, n_2) such that $\varphi(n_1) = \varphi(n_2)$?

[70] By German Arthur **Wieferich**$(1884 - 1954)$. He is also known for his *Wieferich pairs*. **Silverman** proved that ABC conjecture implies that there are infinitely many primes p such that: $p^2 \nmid 2^p - 2$.

[71] Proposed by German mathematician Edmund **Landau**$(1877 - 1938)$ at the International Congress of Mathematicians(1912). Godfrey **Hardy**$(1877 - 1947)$, and John **Littlewood**$(1885 - 1977)$, conjectured that the number of such primes is $\sim c\frac{\sqrt{n}}{log(n)}$ as $n \to +\infty$.

103 The Goormaghtigh conjecture[72] (1917) : The only integer solutions (x, y, m, n) for the equation:

$$\frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1}$$

Are $(5, 2, 3, 5)$ and $(90, 2, 3, 13)$.

104 The Woodall primes[73] (1917) : Are there infinitely many primes of the form

$$n \times 2^n - 1$$

105 The Hardy—Littlewood conjecture(I) (1921) : On the density and distance between the zeros of $\zeta(z)$.

106 The Artin conjecture[74] on the Artin L-functions (1923) : The function $L(\rho, z)$ is entire.

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[72] By Belgian engineer René **Goormaghtigh**$(1893 - 1960)$.

[73] By British mathematician Herbert **Woodall**$(1869 - 1943)$.

[74] It is known that : Artin conjecture $\Longrightarrow$ Dedekind conjecture.

107 The Hardy–Littlewood conjecture[75] (1923) : Let $\pi(x)$ be the number of primes $\leq x$, then

$$\pi(x+y) \;\; \leq \;\; \pi(x)+\pi(y)$$

108 Are there infinitely many sets of consecutive primes of any given pattern ?

109 Are there arbitrarily long arithmetic progressions of consecutive primes.

110 The Artin conjecture[76] on primitive roots(1929) : Let n be not a square and $\neq -1$, then n is a primitive root modulo infinitely many primes p.

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[75] Posed by English mathematicians Godfrey **Hardy**$(1877-1947)$, and John **Littlewood**$(1885-1977)$.

[76] By Austrian mathematician Emil **Martin**$(1898-1962)$. It is the generalization of Gauss conjecture on primitive roots.

111 The Ramanujan–Petersson conjectures[77] (1930).

112 The prime–triplets conjecture[78] : Are there infinitely many prime triplets of the form : $(p, p+2, p+6)$ and $(p, p+4, p+6)$.

113 The Pillai conjecture[79] (1931) : For any n, there is only a finite number of perfect powers (a^k, b^l) such that : $n = a^k - b^l$.

114 The totient problem : Are there infinitely many positive integers n such that

$$\varphi(n) = \varphi(n+1) \quad ?$$

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[77] By German mathematician Hans **Petersson**$(1902 - 1984)$. The early conjectures conjectured by Indian Srinivasa **Ramanujan**$(1887 - 1920)$ in 1916, have been solved.

[78] See Godefrey **Hardy** *: An Introduction to the Theory of Numbers* $(p. 6)$.

[79] By Indian mathematician Subbayya **Pillai**$(1901 - 1950)$, he was killed in a plane crash at the age of 49.

115 The Lehmer problem[80] (1932) : Is there a composite n such that :

$$\varphi(n) | n-1 \quad ?$$

116 The Lehmer conjecture[81] (1933).

117 The Dickson problem[82](1934) : On the periodicity of certain sequence of differences.

118 The Cramér conjecture[83] (1936) :

$$\lim_{n \to +\infty} \frac{p_{n+1} - p_n}{\log^2(p_n)} = 1$$

---

[80] The function φ is the Euler totient function.

[81] Both the problem and conjecture are due to Derrick H. **Lehmer**$(1905 - 1991)$ son of Derrick N. **Lehmer**$(1867 - 1938)$, both were American number theorists.

[82] By American Eugene **Dickson**$(1874 - 1954)$.

[83] By Swedish Harald **Cramér**$(1893 - 1985)$.

119 The Scholz conjecture[84] (1937).

120 The Collatz conjecture[85] (1937).

121 The Luogeng problem[86] (1938) : For every sufficiently big positive integer n, we have the representation

$$n = p_1^k + \cdots + p_s^k.$$

122 The Turán conjecture[87] : Let $(a, b) = 1$ then

$$\pi(n; a, b) > \pi(n; a_1, b)$$

for every $a_1 \not\equiv a[b]$.

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[84] By German Arnold **Scholz**$(1904 - 1942)$.

[85] Known also as the $3n + 1$ *conjecture.* Due to German mathematician Lothar **Collatz**$(1910 - 1990)$.

[86] Known also as Waring–Goldbach problem. It was posed and has been popularized by Chinese Hua **Loogeng**$(1910 - 1985)$ since 1938.

[87] $\pi(n; a, b)$ is the number of primes $p \leq n$ such that $p \equiv a[b]$. Posed by Paul **Turán**$(1910 - 1976)$, a Hungarian mathematician. It is also called the *prime number race*. Numbers like $\pi(n; a, b)$ were introduced by Russian mathematician Pafnuty **Tchebychev**$(1821 - 1894)$.

123 The Erdös$-$Turán conjecture[88] (1941) : The number of ways in which a positive integer n can be written as a sum of k elements from an additive basis B tends to $+\infty$ as n tends to $+\infty$.

124 The Erdös$-$Straus conjecture[89] (1948) : Given an integer $n \geq 2$, has the equation : $\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ any solutions?

125 The Giuca[90] conjecture(1950) : If

$$\sum_{k=1}^{n-1} k^{n-1} + 1 \equiv 0 \ [n]$$

then n is prime.

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[88] In 1964, Paul **Erdös**$(1913 - 1996)$ added the multiplicative version: " a product of k elements". Paul **Turán**$(1910 - 1976)$ was Hungarian mathematician.

[89] Otherwise can the fraction $\frac{4}{n}$ be the sum of 3 unit fractions for any n ?(see Egyptian fraction problem). Ernest **Straus**$(1922 - 1983)$ was an American mathematician.

[90] The converse is due to Polish mathematician Waclaw **Sierpiński**$(1882 - 1969)$. The conjecture is by an Italian mathematician named Giuseppe **Giuga.** There is also a Giuga number and a Giuga sequence.

126 The **Erdös–Moser**[91] problem (1953) : Has the equation

$$1^k + 2^k + 3^k + \cdots + n^k = (n+1)^k$$

solutions other than $1 + 2 = 3$.

127 The **Erdös–Selfridge** conjecture on covering systems[92].

128 Are there infinitely many primes in the sequence of the natural numbers concatenation[93] ?

129 Are there infinitely many primes in the sequence of the primes concatenation ?

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[91] Leo **Moser**(1921 – 1970) was an Austrian mathematician.

[92] John **Selfridge**(1927 – 2010) was an American mathematician.

[93] The concatenation of positive integers is : 12345678910111213 … . By American Charles **Nicol**(1926 – 2013) and John **Selfridge**(1927 – 2010).

130 The p-adic zeta function conjecture : $\zeta_p(k) \neq 0$ when k is an odd positive integer.

131 The Chowla conjecture[94] : Let $(a, b) = 1$, then there are infinitely many consecutive primes (p_n, p_{n+1}) such that :

$$p_n \equiv p_{n+1} \equiv a[b]$$

132 The Erdös conjecture[95] (1955).

133 The Riesel problem[96] (1956) :

Is $509{,}203$ the smallest Riesel number ?

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[94] Posed by Indian mathematician Sarvadaman **Chowla** $(1907 - 1995)$.

[95] Known also under the name : *The minimum overlap problem*.

[96] Due to Swedish mathematician Hans **Riesel** $(1929 - 2014)$. An odd positive number k is a Riesel number if $k2^n - 1$ are composite for all n. See Sierpiński number.

134 Has the equation $\varphi(n) = \sigma(n)$ infinitely many solutions ?

135 The Sierprinski problem[97] : Has the equation $\sigma(n) = \sigma(n+1)$ infinitely many solutions ?

136 The Sierprinski–Schinzel conjecture I : Every $r = \frac{a}{b} \in \mathbb{Q}^+$ may be represented as the quotient of shifted primes

$$r = \frac{p+1}{q+1}$$

137 The Sierprinski–Schinzel conjecture II : There are infinitely many primes p such that : $p = n^2 + (n+1)^2$.

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[97] Due to Polish mathematician Waclaw **Sierpiński**$(1882 - 1969)$.

138 The Schneider problem[98] (1957).

139 The Proth−Gilbreath conjecture[99] (1958) : For every positive k we have :

$$d_1^k = 1$$

140 The Schinzel conjecture[100] (1958) : Let $p_1(x), p_2(x), \ldots$ irreducible polynomials. There are infinitely many integers n such that $p_1(n)$, $p_2(n), \ldots$ are primes.

141 The Schinzel conjecture : Every positive rational number can be written as $\frac{p+1}{q+1}$ and $\frac{p-1}{q-1}$, where p and q are primes.

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[98] In transcendental number theory, also known as : the *four exponentials problem*. It is due to German Theodor **Schneider**(1911 − 1988) who solved the 7^{th} Hilbert problem in 1934.

[99] By French François **Proth**(1852 − 1879) , and American Norman **Gilbreath**(1936−) who rediscovered it in 1958.

[100] Also known as *Schinzel H hypothesis*. By polish mathematician Andrzej **Schinzel**(1937 − 2021). the twin conjecture is a special case.

142 The Newman conjecture[101](1960) : For any k and l, the equation

$$p(n) \equiv k[l]$$

has infinitely many solutions n.

143 The Sierprinski number[102] (1960) : Is $78,557$ the smallest Sierprinski number?

144 The Sierprinski prime : Is $271,129$ the second Sierprinski number ?

145 The Sierprinski prime : Is $271,129$ the smallest Sierprinski prime ?

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[101] Named after American Morris **Newman** $(1924 - 2007)$. The function $p(n)$ is the partition function.

[102] By Polish mathematician Waclaw **Sierpiński**$(1882 - 1969)$. A Sierpiński number is an odd positive integer k such that $k2^n + 1$ is composite for every positive n.

146 The Erdös—Oler conjecture[103] (1961).

147 The Bateman—Horn conjecture[104] (1962).

148 The Leopoldt conjecture[105] in algebraic number theory(1962).

149 The Feit—Thompson conjecture[106](1962) : There are no primes (p, q) such that $\frac{p^q-1}{p-1}$ divides $\frac{q^p-1}{q-1}$.

---

[103] By English Norman **Oler**(1929 – 2011). **Oler** said in his paper : "The problem was suggested to us in the course of conversation by Prof. **Erdös** while he was visiting McGill University".

[104] It was conjectured by Americans Paul **Bateman**(1919 – 2012), and Roger **Horn** (1942–).

[105] Conjectured by German Heinrich **Leopoldt**(1927 – 2011) and it has been established for special cases.

[106] By Americans Walter **Feit**(1930 – 2004) and John **Thompson**(1932–).

150 The Gordan problem[107](1962).

151 The Euclid sequence problem(1963)： Do all primes appear in the sequence[108] with least prime factor

$$1+\prod_{p_1=2}^{p_n} p_i \quad ?$$

152 The Tate conjecture[109](1963).

153 The Rumney problem[110](1963)： Has the equation

$$\sigma(n)+\sigma(m)=\sigma(n+m)$$

infinitely many solutions ?

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[107] By American Basil **Gordan**(1931 – 2012). Known also as the *Gaussian Moat problem*.

[108] Known also as the Euclid –Mullin sequence, after American Albert **Mullin**(1933 – 2017) who posed the problem in 1963.

[109] By American mathematician John **Tate**(1925 – 2019). It is similar in number theory to the Hodge conjecture in algebraic geometry.

[110] Posed by Max **Rumney**.

154 The Grothendieck–Katz conjecture(196?).

155 The Fortune[111] conjecture : $q - p\#$ is a prime or 1.

156 Are there infinitely many repunit[112] primes?

157 The elliptic conjecture : Are there infinitely many integer solutions of elliptic equation : $y^2 = x^3 + ax + b$?

---

[111] By New Zealand Reo **Fortune**(1903 – 1979) who was also a known anthropologist. The number $q - p\#$ is called a *Fortunate number*, q is the smallest prime $> p\#$. The conjecture also says: is there any composite Fortunate number?

[112] Composed by the digit 1 only, like 11 and 111, they can be written in the form $\frac{10^n - 1}{9}$. A repunit is a palindrome number. In many number theory problems, we may consider numbers in any base. In the base 2, repunit numbers are Mersenne numbers M_p.

158 The Siegel conjecture[113](1964) : The density of regular primes is

$$\frac{1}{\sqrt{e}}$$

159 The Shanks conjecture[114](1964).

160 The Gillies conjecture[115] (1964).

161 The Birch–Swinnerton–Dyer conjecture[116](1965).

---

[113] By German mathematician Karl **Siegel**(1896 – 1981).

[114] By American Daniel **Shanks**(1917 – 1996).

[115] By Canadian computer scientist Donald **Gillies**(1928 – 1975).

[116] By English mathematicians Bryan **Birch**(1931 –) and Peter **Swinnerton–Dyer**(1927 – 2018).

162 The Selberg conjecture[117] (1965) : For the eigenvalues of certain integer matrices, there is a lower bound(namely $\frac{1}{4}$).

163 The Wills conjecture[118] (1967).

164 The Lander–Parkin–Selfridge conjecture[119](1967) : Let a_i, b_j be positive integers, then if

$$\sum_{i=1}^{n} a_i^k = \sum_{j=1}^{m} b_j^k$$

then $m + n \geq k$.

165 The Elliot–Halberstam conjecture[120] (1968) .

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[117] Du to Norwegian mathematicians Atle **Selberg**$(1917-2007)$.

[118] By German mathematician Jörg **Wills**$(1937-)$. Also called : the *lonely runner conjecture*.

[119] See the Euler sums conjecture(1752).

[120] Due to American mathematician Peter **Elliott**$(1941-)$ and Czech mathematician Heini **Halberstam**$(1926-2014)$.

166 The Grimm conjecture[121] (1969) : If $a_1, \dots, a_k$ are consecutive composite integers, then ∃ different primes p_i such that : $p_i | a_i$.

167 The odd super−perfect[122] numbers(1969) : Are there any odd super-perfect number ?

168 The even super−perfect numbers (1969) : Are there infinitely many even super-perfect numbers ?

169 Are there any almost−perfect numbers(other than 2^k) and quasi−perfect numbers[123] ?

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[121] Posed by American Carl **Grimm**$(1926 - 2018)$.

[122] A positive integer n is *super-perfect* if : $\sigma(\sigma(n)) = 2n$, where σ is the sum of the divisors of n. It was defined by Indian mathematician D. **Suryanarayana**(1969).

[123] A positive integer n is *almost-perfect* if : $\sigma(n) = 2n - 1$, all 2^k are almost –perfect. A positive integer n is *quasi-perfect* if : $\sigma(n) = 2n + 1$.

170 The Langlands program[124] (1969).

171 The Schanuel conjecture[125](196?) :

$$trdeg\,[\,\mathbb{Q}(\alpha_1,\dots,\alpha_n,e^{\alpha_1},\dots,e^{\alpha_n})] \geq n\ \ .$$

172 The Hall conjecture[126] (1970) : Let x, y be two integers such that $x^2 \neq y^3$ then : $|x^2 - y^3| > c\sqrt{|x|}$.

173 The Greenberg conjecture[127] I (1971).

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[124] It is a collection of conjectures on analytic functions in number theory and related geometric objects, posed by Canadian mathematician Robert **Langlands**(1936−).

[125] By American Stephen **Schanuel** (1933 − 2014). Seen as a continuation of Weierstrass−Lindemann theorem, it has consequences in the theory of transcendental numbers.

[126] By American mathematician Marshall **Hall**(1910 − 1990).

[127] Proposed by American mathematician Ralph **Greenberg**(1944−).

174 The Singmaster conjecture[128] (1971) : On the numbers in Pascal triangle.

175 The Stark conjecture[129](1971) : The leading coefficient in the Taylor expansion of the Artin L −function is the product of a regulator and an algebraic number.

176 The Montgomery conjecture[130] (1973) : The pair correlation function $f(x)$ between the nontrivial roots of ζ function is asymptotic to

$$1-\left(\frac{sin(\pi x)}{\pi x}\right)^2$$

177 The P−S−W primality testing conjecture[131] .

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[128] Due to American mathematician David **Singmaster**(1938 − 2023).

[129] Due to American mathematician Harold **Stark** (1939−)There is a special case of it, known as the Brumer−Stark conjecture.

[130] Due to American mathematician Hugh **Montgomery**(1944 −).

[131] Due to Americans Carl **Pomerance**(1944−), John **Selfridge**(1927 − 2010) and Samuel **Wagstaff**(1945−).

178 The **Selfridge** conjecture[132] : The number $g(n)$ of distinct prime factors of Fermat numbers F_n is not montonic.

179 Are there infinitely many cluster[133] primes?

180 The **Silverman** problem[134] : Consider the product

$$S_m = \prod_{n=1}^{m} \frac{p_n + 1}{p_n - 1}$$

Does the sequence S_m contain integers when $m > 8$?

181 The irrationality measure problem.

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[132] Due to American John **Selfridge**$(1927 - 2010)$.

[133] A cluster prime p is a prime that for which every even integer $< p - 2$ is the difference of two primes $\leq p$. Posed by John **Selfridge**$(1927 - 2010)$.

[134] By mathematician David **Silverman**.

182 The Lang conjecture[135] : the inequality

$$\left|x-\frac{a}{b}\right|<\frac{1}{b^2 log^{1-\epsilon}(b)}$$

has only infinitely many rational solutions $\frac{a}{b}$.

183 The Erdös conjecture (1976) : If E is a set of positive numbers n such that $\sum \frac{1}{n}=+\infty$ then E contains arbitrary long arithmetic progressions.

184 The solitary[136] number problem.

185 The Anderson$-$Hickerson conjecture on solitary numbers (1977).

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[135] Due to French mathematician Serge **Lang** $(1927-2005)$.

[136] A solitary number is not friendly. All primes and prime powers are solitary. We do not know in general how to prove that a number is solitary.

186 The Grandall conjecture[137] on the $aX + b$ problem (1978).

187 The Erdös conjecture (1979) : The ternary expansion of 2^n omits the digit 2 for every $n \geq 9$.

188 The Baillie−PSW test problem[138] (1980).

189 The Szpiro conjecture[139] (1981) :

$$|\Delta| \leq C(\epsilon) \times f^{6+\epsilon}.$$

---

[137] It is an extension of the Collatz conjecture (1937).

[138] Robert **Baillie** i an American mathematician.

[139] By French mathematician Lucien **Szpiro** $(1941 - 2020)$.

190 The Firrozbakht conjecture[140](1982) :

$$p_{n+1} \leq {p_n}^{1+\frac{1}{n}}$$

191 The ABC conjecture[141] (1985) : Let a, b, c positive integers, then

$$c \leq rad(abc)^{1+\epsilon}$$

192 The Beilinson conjectures[142](1985).

193 Are there infinitely many reversible[143] primes like :

$$13-31, \quad 17-71 \ , \ 167-761, \ldots$$

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[140] By Iranian mathematician فريدة فيروز بخت(1962 – 2019). She taught mathematics at the university of Isfahan.

[141] The function $rad(n)$ is the product of distinct prime factors of n, so we have always $rad(n) \leq n$. This has been posed by French Joseph **Oesterlé**(1954–) and English David **Masser**(1948–) around the year 1985. So it is also known as Oesterlé–Masser conjecture.

[142] Due to American mathematician Alexander **Beilinson**(1957–). They are Linked to Milnor conjecture(1970) and Bloch–Kato conjecture(1978), both proved by Russian Vladimir **Voevodsky**(1966 – 2017).

[143] A *reversible* number is a number that remains the same when read in the two senses, there are infinitely many of them by construction. It may be named *palindromic* number. There are also palindromic words and expressions in natural languages as : *noon* and *nurses run*.

194 Are there infinitely many deletable primes?

195 The Erdös–Mollin–Walsh conjecture (1986) : There are no three consecutive powerful numbers.

196 The Andrica conjecture[144] (1986) : For any n we have

$$\sqrt{p_{n+1}} - \sqrt{p_n} \leq 1$$

197 Is :

$$\lim_{n \to +\infty} \left(\sqrt{p_{n+1}} - \sqrt{p_n}\right) = 0 \quad ?$$

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[144] By Romanian mathematician Dorin **Andrica**, he teaches mathematics at the faculty of Mathematics and Computer Sciences(Romania).

198 The Vojta conjecture[145] (1987).

199 The Don Zagier conjectures[146] (1988).

200 The Wagstaff primes[147] (1989) : Are there infinitely many primes of the form $\frac{2^p+1}{3}$.

201 The Bateman[148] conjecture (1989) : Let M_p be a Mersenne number, if $p = 2^k \pm 1$ or $p = 4^k \pm 3$ and M_p is prime, then : $\frac{2^p+1}{3}$ is a prime.

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[145] By American Paul **Voita**(1957−).

[146] By German number theorist Don **Zagier**(1951−) , they are on the values of the Dedekind zeta function.

[147] By American Samuel **Wagstaff**(1945−).

[148] It was conjectured by Americans : Paul **Bateman**(1919 − 2012), John **Selfridge**(1927 − 2010) and Samuel **Wagstaff**(1945−). It is also known as the "*New Mersenne Conjecture*".

202 Are there any Wall−Sun−Sun primes[149] ? (1992).

203 The Pickover[150] problem(1992).

204 The Beal conjecture[151] (1993).

205 The Browkin−Brzeziński problem[152] (1994) ： It generalizes the ABC conjecture.

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[149] By American Donald **Wall**(1921 − 2000), Chinese Sun **Zhihong**(1965 −) and Sun **Zhiwei**(1965 −).

[150] By Clifford **Pickover**(1957−) , it is similar to the Collatz conjecture.

[151] After English Andrew **Wiles**(1953−) proved **Fermat** Last Theorem(1993), American businessman and amateur Daniel **Beal** (1952−) conjectured that the similar equation $x^k + y^l = z^m$ has also no integers solutions (x, y, z), then in 1997 he gave 50000 $ bounty for its solution.

[152] Known also as the *N conjecture*.

206 Are there distinct primes (m, n) such that[153] : $m\sigma_2(m) = n\sigma_2(n)$.

207 Let $k \geq 3$, is the numbers

$$\sum_{n\geq 1} \frac{\sigma_k(n)}{n!}$$

irrational ?

208 The McIntosh conjecture[154] (1995) : there are infinitely many Wolstenholme prime, i.e. primes p that verify

$$\binom{2p-1}{p-1} \equiv 1[p^4]$$

209 The Casas–Alvero conjecture[155] (2001).

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[153] $\sigma_2(n)$ is the sum of the square powers of the divisors of n.

[154] See the Wolstenholme problem.

[155] Posed by Spanish Eduardo **Casa**–**Alvero**$(1948-)$, he teaches mathematics at the University of Barcelona.

210 The Agrawal conjecture[156] (2002).

211 The Redmond−Sun conjecture[157] (2006) : Every interval $[a^n, b^m]$, where $a, b, n, m \geq 2 \in \mathbb{N}$, contains primes numbers with known finite exceptions.

212 Are there any two algebraic numbers α, β such that $log(\alpha)$ and $log(\beta)$ are algebraically independent ?

213 Algebraic independence[158] of the numbers π and e.

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[156] Du to Indian Manindra **Agrawal** (1966 −).

[157] By Chinese Sun **Zhiwei**(1965 −) and American Stephen **Redmond**. The Catalan conjecture(solved in 2002) and the Legendre conjecture are special cases.

[158] It is implied by the Schanuel conjecture.

214 Are the numbers π and e normal[159] ?

215 Are the following numbers[160] transcendental (or at least irrational) :

$$\frac{e}{\pi}\ , \qquad e\pi\ , \qquad e+\pi\ , \qquad e-\pi$$

216 Are the following numbers[161] transcendental (or at least irrational) :

$$\pi^{\pi}\ , \qquad e^{e}\ , \qquad \pi^{e}.$$

---

[159] A real number is *normal* if its digits are evenly distributed. All rational numbers are non-normal.

[160] It is evident that if $\frac{e}{\pi}$ is transcendental then so is $\frac{\pi}{e}$. At least one of the two numbers $e+\pi$ and $e-\pi$ is irrational. Swiss Johann **Lambert** $(1728-1777)$ was the first to prove that π is irrational. In 1873, French Charles **Hermite**$(1822-1901)$ prove that e is is transcendental. In 1882 , German Carl **Lindeman** $(1852-1939)$ proved that π is transcendental.

[161] In 1934, Russian Alexander **Gelfond**$(1906-1968)$ and independently German Theodor **Schneider**$(1911-1988)$ both proved that the number e^{π} is transcendental (so irrational).

217 Are the following numbers transcendental (or at least irrational) :

$e^{\sqrt{2}}$, $\pi^{\sqrt{2}}$, $a^{\pi}\ (a \neq 0,1,e)$, $a^{e}\ (a \neq 0,1)$, $log(e\pi)$[162]

$$\Gamma\left(\frac{1}{5}\right).$$

218 Are the numbers

$$\pi^{\pi^{\pi^{\cdot^{\cdot^{\cdot}}}}}$$

transcendental (or at least irrational) for every exponent of π ?

219 Are the numbers

$$e^{e^{e^{\cdot^{\cdot^{\cdot}}}}}$$

transcendental (or at least irrational) for every exponent of e ?

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[162] Or simply $log(\pi)$.

220 Are there infinitely many balanced[163] primes ?

221 The Lynchrel[164] problem (2011).

222 The Greenberg conjecture[165] II (2016).

223* Let n a composite odd integer, when[166] does : $gcd\big(n,\varphi(n)\big) = 1$?
Are there infinitely many n : $gcd\big(n,\varphi(n)\big) = 1$?

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[163] A balanced prime p_n is a prime of the form $p_n = \frac{p_{n-1}+p_{n+1}}{2}$.

[164] A Lynchrel number cannot be a palindromic. The word *Lynchrel* is a deformed person's name, coined by American Wade Van **Landingham**.

[165] Due to American Ralph **Greenberg**$(1944-)$.

[166] If n a prime, $gcd\big(n,\varphi(n)\big) = 1$. If n is even, $gcd\big(n,\varphi(n)\big) \geq 2$.

224* What is the value of

$$\lim_{n \longrightarrow \infty} \sigma(\sigma(\ldots \sigma(n)))$$

225* Exclude n from $\sigma(n)$: What is the value of

$$\lim_{n \longrightarrow \infty} \sigma(\sigma(\ldots \sigma(n)))$$

226* Consider the sequence : $s_n = \sum_{i=1}^{n} p_i$. Are there infinity many primes in s_n ?

227* Are there infinitely many squares in the sequences : $s_n = \sum_{i=1}^{n} p_i$?

228* Let p_i, q_j be primes. Find[167] two analytic functions $\boldsymbol{F}$ and $L(z,\chi)$ such that

$$\sum_{2n>p_i\geq n}\ \sum_{q_j\leq n}\frac{1}{(n-p_i)^z+(n-q_j)^z}=\boldsymbol{F}(L(z,\chi))$$

229* Are there infinitely many squares[168] of the form $p\ \#\ -1$?

230* Let $N\geq 10$ an even positive integer[169] , then $N=2\sum_1^k p_i$.

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[167] The series is supposed to have a link with the Goldbach conjecture. When $n\to+\infty$,there is a problem of convergence, and the number of singularities of $\boldsymbol{F}(L(z,\chi))$ increases.

[168] It is easy to show that $p\ \#\ +1$ cannot be a square. In 1975 , Paul **Erdös**$(1913-1996)$ and John **Selfridge**$(1927-2010)$ proved that the product of consecutive integers cannot be a power.

[169] For example : $12=\ 2(3+3)$, $14=2(2+5)$, $16=\ 2(3+5)$, $20=\ 2(2+3+5)$. When $k=1$, it does correspond to the Goldbach's conjecture(in the case $n=2p$).

231* Infinitude of primes of the form[170]: $n! - n - 1$.

232* Infinitude of primes of the form : $n! + n + 1$.

233* Infinitude of prime couples of the form :

$$(\ n! - n - 1 \ , \qquad n! + n + 1 \ )$$

Open Problems

in

Number Theory

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[170] Since the numbers $n! \pm k$ are composite for $2 < k \leq n$, it is highly probable that the numbers $n! + n + 1$ and $n! + n - 1$ could be prime, and perhaps for infinitely many times.